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Xavier Bekaert,[a] **Euihun Joung**[b] **and Jihad Mourad**[b]

[a]*Laboratoire de Mathématiques et Physique Théorique,*
Unité Mixte de Recherche 6083 du CNRS, Fédération Denis Poisson,
37200 Tours, France

[b]*AstroParticule et Cosmologie,*
Unité Mixte de Recherche 7164 du CNRS,
Université Paris VII, Bâtiment Condorcet 75205 Paris Cedex 13, France

E-mail: bekaert@lmpt.univ-tours.fr, joung@apc.univ-paris7.fr,
mourad@apc.univ-paris7.fr

Abstract: Cubic couplings between a complex scalar field and a tower of symmetric tensor gauge fields of all ranks are investigated. A symmetric conserved current, bilinear in the scalar field and containing r derivatives, is provided for any rank $r \geqslant 1$ and is related to the corresponding rigid symmetry of Klein-Gordon's Lagrangian. Following Noether's method, the tensor gauge fields interact with the scalar field via minimal coupling to the conserved currents. The corresponding cubic vertex is written in a compact form by making use of Weyl's symbols. This enables the explicit computation of the non-Abelian gauge symmetry group, the current-current interaction between scalar particles mediated by any gauge field and the corresponding four-scalar elastic scattering tree amplitude. The exact summation of these amplitudes for an infinite tower of gauge fields is possible and several examples for a definite choice of the coupling constants are provided where the total amplitude exhibits fast (e.g. exponential) fall-off in the high-energy limit. Nevertheless, the long range interaction potential is dominated by the exchange of low-spin ($r \leqslant 2$) particles in the low-energy limit.

Contents

1 Introduction and summary of results

The role of higher-spin fields in fundamental interactions is still unclear. On the one hand, starting from spin two, the potential coupling constants have negative mass dimensions leading to power counting nonrenormalisable theories. On the other hand, higher spin particles have a crucial role in the softness of string interactions at high energies; the infinite tower of massive higher-spin states provides a regularisation in the ultraviolet. Confronting this fact and the example of Vasiliev's theory [1] (reviewed e.g. in [2]) with the many no-go

theorems [3–6] involving a finite number of massless higher-spin fields suggests that an infinite collection of higher-spin fields is a necessary ingredient for building a consistently interacting theory. Furthermore, the algebra of higher-spin symmetries is expected to have consistent Lorentz covariant truncations only for gauge fields with spin not greater than two.

Here, we would like to examine the issues of the high energy behaviour and the gauge symmetries in more details in the framework of a simple example: the cubic couplings between a matter scalar field and a collection of higher-spin gauge fields. The model is consistent from quadratic order in the gauge and matter fields up to cubic couplings of two scalar and one gauge field. This model can be used to reliably calculate tree level amplitudes for the elastic scattering of two scalar particles. It also gives a hint on the non-Abelian generalization of the gauge algebra, in our case it is the algebra of unitary operators on $L^2(\mathbb{R}^n)$ where n is the spacetime dimension.

Let us first describe briefly the model. We start with a free matter scalar field $|\phi\rangle$, with the Klein-Gordon action

$$S_0[\phi] = -\langle\phi|\hat{P}^2 + m^2|\phi\rangle. \tag{1.1}$$

It gives rise to an infinite set of conserved Noether currents. Alternatively, the generating function

$$\phi(x - q/2)\,\phi^*(x + q/2) = \sum_r \frac{1}{r!} \overset{(r)}{J}_{\mu_1...\mu_r}\, q^{\mu_1}\ldots q^{\mu_r}, \tag{1.2}$$

when expanded in the auxiliary variable u gives as coefficients symmetric conserved currents $J^{(r)}$ which are improved Noether currents. The higher-spin gauge fields $h^{(r)}$ can also be grouped in a generating function

$$h(x, p) = \sum_{r \geqslant 0} \frac{1}{r!} \overset{(r)}{h}_{\mu_1...\mu_r}(x)\, p^{\mu_1}\ldots p^{\mu_r}$$

which we interpret as defining a Hermitian operator $\hat{H}$ acting on the scalar field. The currents allow minimal couplings with the higher-spin fields. The first important remark is that the sum of the cubic couplings takes the simple form

$$S_1[\phi, h] = -\langle\phi|\hat{H}|\phi\rangle. \tag{1.3}$$

The precise relationship between the operator $\hat{H}$ and the generating function $h(x, p)$ for higher-spin field $h^{(r)}$ is that the latter is the Weyl symbol of the former. Basic facts about Weyl calculus are recalled in appendix A. Denoting $\hat{G} = \hat{P}^2 + m^2 + \hat{H}$, the action $S_0 + S_1$, which is clearly of the form $-\langle\phi|\hat{G}|\phi\rangle$, is invariant under the unitary transformations

$$|\phi\rangle \rightarrow \hat{U}|\phi\rangle, \tag{1.4}$$

provided $\hat{G}$ transforms as

$$\hat{G} \rightarrow \hat{U}\,\hat{G}\,\hat{U}^{-1}. \tag{1.5}$$

The second important observation is that this transformation reduces to lowest order in $\hat{H}$ to the gauge transformation of massless higher-spin fields.[1] This is shown in section 3.

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[1] Similar ideas on the link between Weyl quantisation and Noether couplings between matter and gauge fields have been pushed forward previously in the context of conformal higher-spin theory by Segal [7]. Symbol calculus made one of its earliest appearance in the subject of higher spin interactions in the construction of higher-spin (super)algebras [8].

We next consider tree level scattering amplitudes which can be easily calculated in our framework. The model gives rise to cubic vertices, which together with the propagators of the higher-spin fields allow the calculation of the tree amplitudes for the scattering of two scalar particles. The propagators which are suitable for our purposes were found in [9] where no assumption about the vanishing of the double trace of the fields were made. One may ask about the coupling constants of the theory. In fact, there is an infinite number of them, which are hidden in the correspondence between $\hat{H}$ and the higher-spin fields $h^{(r)}$ or, by a field redefinition, in the kinetic terms of $h^{(r)}$. We have one coupling constant λ with dimension of length and a collection of dimensionless couplings a_r associated with each spin r. In fact all these dimensionless couplings can be grouped in a generating function $a(z)$

$$a(z) = \sum_{r=0}^{\infty} \frac{a_r}{r!} z^r. \qquad (1.6)$$

We will show that the tree level amplitude of the two-scalar scattering $\phi\,\phi \to \phi\,\phi$ and the non-relativistic potential can both be expressed simply in terms of this generating function. Its behavior near the origin determines the static interaction potential and its behavior at large negative arguments determines the high energy scattering amplitudes. The explicit expression of the scattering amplitude turns out to be very simple and is given, in terms of the Mandelstam variables, by

$$A(s,t,u) = -\frac{\lambda^{-2}}{t} \left[a\left(-\frac{\lambda^2}{8}\left(\sqrt{s}+\sqrt{-u}\right)^2\right) + a\left(-\frac{\lambda^2}{8}\left(\sqrt{s}-\sqrt{-u}\right)^2\right) - a_0 \right]. \qquad (1.7)$$

It can be very soft at high energies if the function a is small for large negative argument. The static potential due to the exchange of a spin r particle between two mass m particles with interdistance $\vec{x}$ can be deduced and is given by

$$\overset{(r)}{V}(\vec{x}) = \frac{a_r}{4\,r!} \left(-\frac{(m\lambda)^2}{2}\right)^{r-1} \frac{1}{4\pi\,|\vec{x}|}. \qquad (1.8)$$

If λ is of the order of the Planck length and m of the proton mass, then $(m\lambda) \ll 1$ and the potentials for higher spins are negligible with respect to the Newtonian one provided the coefficients a_r do not grow fastly with r. Unitarity leads to positive coefficients a_r but otherwise the generating function is arbitrary within our framework. We expect higher order consistency to further constrain this function.

The plan of the paper is as follows: section 2 presents a concise reformulation of the so-called Noether method for introducing consistent interactions between matter and gauge fields in terms of various generating functions. This formalism together with Weyl calculus is applied in section 3 to the construction of the cubic vertices that are bilinear in a complex scalar field and linear in a tensor gauge field. Section 4 is devoted to the four-scalar elastic scattering tree amplitude due to the exchange of a single tensor gauge field. It is expressed in terms of Chebyshev's or Gegenbauer's polynomials. The high-energy behaviour of their sum, corresponding to an infinite tower of exchanged tensor gauge fields, is discussed in section 5. The non-relativistic interaction potential is obtained

and discussed in section 6. The paper ends with a short conclusion in section 7 and several appendices:[2] appendix A is a brief introduction to the formulation of quantum mechanics in terms of Weyl symbols [10]. In appendix B, the Mandelstam variables and various limits of elastic scattering are recalled. Appendix C contains the definitions and some useful formulas for Chebyshev's or Gegenbauer's polynomials.

2   Generating functions and the Noether method

A *symmetric conserved current* of rank $r \geqslant 1$ is a real contravariant symmetric tensor field $J^{(r)}$ obeying the conservation law

$$\partial_{\mu_1} \overset{(r)}{J}{}^{\mu_1 \ldots \mu_r}(x) \approx 0 \,.$$

where the "weak equality" symbol $\approx$ stands for "equal on-mass-shell," i.e. modulo terms proportional to the Euler-Lagrange equations. A *generating function of conserved currents* is a real function $J(x, u)$ on phase space which is (i) a formal power series in the "momenta" u_μ and (ii) such that

$$\left(\frac{\partial}{\partial u_\mu} \frac{\partial}{\partial x^\mu} \right) J(x, u) \approx 0 \,. \tag{2.1}$$

The terminology follows from the fact that all the coefficients of order $r \geqslant 1$ in the power expansion of the generating function

$$J(x, u) = \sum_{r \geqslant 0} \frac{1}{r!} \overset{(r)}{J}{}^{\mu_1 \ldots \mu_r}(x) \, u_{\mu_1} \ldots u_{\mu_r} \tag{2.2}$$

are all symmetric currents which from eq. (2.1) are conserved.

A *symmetric tensor gauge field* of rank $r \geqslant 1$ is a real covariant symmetric tensor field $h^{(r)}$ whose gauge transformations are [11]

$$\delta_\varepsilon \overset{(r)}{h}{}_{\mu_1 \ldots \mu_r}(x) = r \, \partial_{(\mu_1} \overset{(r-1)}{\varepsilon}{}_{\mu_2 \ldots \mu_r)}(x) + \mathcal{O}(h) \,, \tag{2.3}$$

where the gauge parameter $\varepsilon^{(r-1)}$ is a covariant symmetric tensor field of rank $r-1$ and the round bracket denotes complete symmetrisation with weight one. For lower ranks $r = 1$ or 2, the transformation (2.3) either corresponds to the U(1) gauge transformation of the vector ($r = 1$) gauge field or to the linearised diffeomorphisms of the metric ($r = 2$). By comparison with the spin-two case, this formulation of higher-spin gauge fields is sometimes called "metric-like" (in order to draw the distinction with the "frame-like" version where the gauge field is not completely symmetric). A *generating function of gauge fields* is a real function $h(x, v)$ on configuration space (i) which is a formal power series in the velocities v^μ and (ii) whose gauge transformations are

$$\delta_\varepsilon h(x, v) = \left(v^\mu \frac{\partial}{\partial x^\mu} \right) \varepsilon(x, v) + \mathcal{O}(h) \,, \tag{2.4}$$

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[2]Except in appendix A, we set $\hbar = c = 1$.

where $\varepsilon(x,v)$ is also a formal power series in the velocities. The nomenclature follows from the fact that all the coefficients of order $r \geqslant 1$ in the power expansion of the generating function

$$h(x,v) = \sum_{r \geqslant 0} \frac{1}{r!} \overset{(r)}{h}_{\mu_1 \ldots \mu_r}(x) \, v^{\mu_1} \ldots v^{\mu_r} \tag{2.5}$$

are all symmetric tensor gauge fields due to (2.4) with

$$\varepsilon(x,v) = \sum_{r \geqslant 0} \frac{1}{r!} \overset{(r)}{\varepsilon}_{\mu_1 \ldots \mu_r}(x) \, v^{\mu_1} \ldots v^{\mu_r} \, .$$

Of course, in the context of Noether couplings, the "velocities" v^μ and "momenta" u_ν are interpreted as mere auxiliary variables. Let us introduce a nondegenerate bilinear pairing $\ll \| \gg$ between the generating functions $J(x,u)$ and $h(x,v)$ on the configuration and phase spaces respectively,

$$\ll J \| h \gg := \sum_{r \geqslant 0} \int d^n x \, \big\langle \, \overset{(r)}{J}(x) \, , \, \overset{(r)}{h}(x) \, \big\rangle \, , \tag{2.6}$$

where $\big\langle \, J^{(r)}(x) \, , \, h^{(r)}(x) \, \big\rangle$ is the contraction between the current and the gauge field:

$$\big\langle \, \overset{(r)}{J}(x) \, , \, \overset{(r)}{h}(x) \, \big\rangle = \frac{1}{r!} \overset{(r)}{J}{}^{\mu_1 \ldots \mu_r}(x) \, \overset{(r)}{h}_{\mu_1 \ldots \mu_r}(x) \, . \tag{2.7}$$

This bilinear form can be written in terms of the generating functions as

$$\ll J \| h \gg = \int d^n x \, \exp \left(\frac{\partial}{\partial u_\mu} \frac{\partial}{\partial v^\mu} \right) J(x,u) \, h(x,v) \Big|_{u=v=0} \, . \tag{2.8}$$

Let us denote by $\ddagger$ the adjoint operation for the pairing (2.8) in the sense that

$$\ll J \| \hat{\hat{O}} \, h \gg = \ll \hat{\hat{O}}^\ddagger J \| h \gg \, ,$$

where $\hat{\hat{O}}$ is an operator acting on the vector space of functions on configuration space (the double hat stands for "second quantisation" in the sense that the operator acts on symbols of "first quantised" operators). Notice that $(v^\mu)^\ddagger = \partial/\partial u_\mu$ and $(\partial/\partial x^\mu)^\ddagger = -\partial/\partial x^\mu$ imply the useful relation

$$\left(v^\mu \frac{\partial}{\partial x^\mu} \right)^\ddagger = - \left(\frac{\partial}{\partial u_\mu} \frac{\partial}{\partial x^\mu} \right) \, . \tag{2.9}$$

The *matter action* is a functional $S_0[\phi]$ of some matter fields collectively denoted by ϕ. The Euler-Lagrange equations of these matter fields is such that there exists some conserved current $J^{(r)}[\phi]$. The Noether method for introducing interactions is essentially the "minimal" coupling between a gauge field $h^{(r)}$ and a conserved current $J^{(r)}[\phi]$ of the same rank. Accordingly, the *Noether interaction* between gauge fields and conserved currents is the functional of both matter and gauge fields defined as the pairing between the generating functions

$$S_1[\phi,h] := - \ll J \| h \gg \, . \tag{2.10}$$

Let us assume that there exists a gauge invariant action $S[\phi, h]$ whose power expansion in the gauge fields starts as follows

$$S[\phi, h] = S_0[\phi] + S_1[\phi, h] + S_2[\phi, h] + \mathcal{O}(h^3).$$ (2.11)

The variation of the Noether interaction (2.10) under (2.4)

$$\delta_\varepsilon S_1[\phi, h] = - \ll J \| \delta_\varepsilon h \gg + \mathcal{O}(h),$$

is at least of order one in the gauge fields when the equations of motion for the matter sector are obeyed,

$$\delta_\varepsilon S_1[\phi, h] \approx \mathcal{O}(h),$$ (2.12)

because the properties (2.1) and (2.9) imply that

$$\ll J \| \left(v \cdot \frac{\partial}{\partial x} \right) \varepsilon \gg = - \ll \left(\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial x} \right) J \| \varepsilon \gg \approx 0.$$ (2.13)

Actually, the crucial property (2.12) works term by term since

$$\int d^n x \, \overset{(r)}{J}{}^{\mu_1 \dots \mu_r}(x) \, \partial_{\mu_1} \overset{(r-1)}{\varepsilon}{}_{\mu_2 \dots \mu_r}(x) = - \int d^n x \, \partial_{\mu_1} \overset{(r)}{J}{}^{\mu_1 \dots \mu_r}(x) \, \overset{(r-1)}{\varepsilon}{}_{\mu_2 \dots \mu_r}(x) \approx 0.$$

The equation (2.12) implies that the action (2.11) is indeed gauge-invariant at lowest order in the gauge fields because the terms that are proportional to the Euler-Lagrange equations $\delta S_0 / \delta \phi$ of the matter sector can be compensated by introducing a gauge transformation $\delta_\varepsilon \phi$ of the matter fields, independent of the gauge fields h and linear in the matter fields ϕ, such that

$$\delta_\varepsilon \left(S_0[\phi] + S_1[\phi, h] \right) = \mathcal{O}(h).$$ (2.14)

A *Killing tensor field* of rank $r - 1 \geqslant 0$ on $\mathbb{R}^n$ is a covariant symmetric tensor field $\overset{(r-1)}{\overline{\varepsilon}}$ solution of the generalised Killing equation

$$\partial_{(\mu_1} \overset{(r-1)}{\overline{\varepsilon}}{}_{\mu_2 \dots \mu_r)}(x) = 0.$$

A *generating function of Killing fields* is a function $\overline{\varepsilon}(x, v)$ on configuration space which is (i) a formal power series in the velocities and (ii) such that $\overline{\varepsilon}(x + v\tau, v) = \overline{\varepsilon}(x, v)$ for any τ. Then the coefficients in the power series

$$\overline{\varepsilon}(x, v) = \sum_{r \geqslant 0} \frac{1}{r!} \overset{(r)}{\overline{\varepsilon}}{}_{\mu_1 \dots \mu_r}(x) \, v^{\mu_1} \dots v^{\mu_r}$$

are all Killing tensor fields on $\mathbb{R}^n$. The variation (2.3) of the gauge field vanishes if the gauge parameter is a Killing tensor field. Therefore the corresponding gauge transformation $\delta_{\overline{\varepsilon}} \phi$ of the matter fields is a rigid symmetry of the matter action $S_0[\phi]$:

$$\delta_{\overline{\varepsilon}} S_0[\phi] = - \delta_{\overline{\varepsilon}} S_1[\phi, h] \big|_{h=0} = 0,$$

due to (2.14) and the fact that $\delta_\varepsilon\phi$ is independent of the gauge fields. In turn, this shows that the conserved current $J^{(r)}[\phi]$ must be equal, on-shell and modulo a trivial conserved current (sometimes called an "improvement"), to the Noether current associated with the latter rigid symmetry of the action $S_0[\phi]$. A careful look at the one-to-one correspondence between equivalence classes of rigid symmetries of the action and conserved currents provided by Noether's theorem (see e.g. the section 2 of [12] for a concise review) allows to prove also the following fact: if the Noether interaction is translation invariant (i.e. the Noether current does not depend on x) then the corresponding rigid symmetry of the matter field does not depend on x.

3 Minimal coupling of a scalar to higher-spin gauge fields

3.1 Conserved current of any rank from scalar action

Consider a matter sector made of a free complex scalar field ϕ, of mass square $m^2 \geqslant 0$, propagating on Minkowski spacetime with mostly plus metric $\eta_{\mu\nu}$. The matter action is the quadratic functional

$$S_0[\phi] = -\int d^n x \left(\eta^{\mu\nu} \partial_\mu \phi^*(x) \partial_\nu \phi(x) + m^2 \phi^*(x) \phi(x) \right), \qquad (3.1)$$

which gives an Euler-Lagrange equation: $(\Box - m^2)\phi(x) \approx 0$. Now consider the following function with an auxiliary variable q^μ

$$\breve{\rho}(x,q) := \phi^*(x - q/2)\, \phi(x + q/2)\,. \qquad (3.2)$$

It obeys a conservation law,

$$\left(\eta^{\mu\nu} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial x^\nu} \right) \breve{\rho}(x,q) = \phi^*(x - q/2)\partial^2\phi(x + q/2)$$
$$- \partial^2\phi^*(x - q/2)\,\phi(x + q/2) \approx 0\,, \qquad (3.3)$$

and can be considered as a generating function for symmetric conserved currents. Notice that eq. (3.3) is similar to eq. (2.1) except that the metric must be used.[3] Therefore one finds that a very simple generating function of conserved currents is $J(x,u) = \breve{\rho}(x,-i\,u)$ where the factor i has been introduced in such a way that the function is real. It can be formally written in terms of the wave function $\phi(x)$ as

$$J(x,u) = \phi^*(x + i\,u/2)\,\phi(x - i\,u/2) = |\,\phi(x - i\,u/2)\,|^2\,. \qquad (3.4)$$

where reality is manifest. The condition (2.1) can again be checked by a direct computation.

Moreover, the Taylor expansion of $J(x,u)$ in power series of u^μ:

$$J(x,u) = \sum_{r=0}^{\infty} \frac{1}{r!} \overset{(r)}{J}{}_{\mu_1\ldots\mu_r}(x)\, u^{\mu_1}\ldots u^{\mu_r}, \qquad (3.5)$$

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[3]The Minkowski metric provides an isomorphism between the tangent and cotangent spaces via the identification $u^\mu = \eta^{\mu\nu}u_\nu$, which induces an isomorphism between the spaces of functions on the configuration and phase spaces.

leads to the explicit expression of the symmetric conserved currents

$$\overset{(r)}{J}_{\mu_1...\mu_r}(x) = \left(\frac{i}{2}\right)^r \sum_{s=0}^{r} (-1)^s \binom{r}{s} \partial_{(\mu_1} \ldots \partial_{\mu_s} \phi(x) \, \partial_{\mu_{s+1}} \ldots \partial_{\mu_r)} \phi^*(x) \,, \qquad (3.6)$$

where all indices of the currents have been lowered because its explicit expression is in terms of derivatives of the scalar field. These currents are proportional to the ones already introduced in [13]. Various explicit sets of conserved currents were also provided in [14]. The symmetric conserved current (3.6) or rank r is bilinear in the scalar field and contains exactly r derivatives. The currents of any rank are real thus, if the scalar field is real then the odd rank currents are absent due to the factor in front of (3.6). Notice that the symmetric conserved current of rank two

$$\overset{(2)}{J}_{\mu\nu}(x) = -\frac{1}{4} \left(\partial_\mu \partial_\nu \phi^*(x) \, \phi(x) + \phi^*(x) \, \partial_\mu \partial_\nu \phi(x) - 2 \, \partial_{(\mu} \phi^*(x) \, \partial_{\nu)} \phi(x) \right) \,, \qquad (3.7)$$

is distinct from the canonical energy-momentum tensor

$$T_{\mu\nu}(x) = \partial_{(\mu} \phi^*(x) \, \partial_{\nu)} \phi(x) - \frac{1}{2} \, \eta_{\mu\nu} \left(|\partial \phi(x)|^2 + m^2 |\phi(x)|^2 \right) \,,$$

though, on-shell they differ only from a trivially conserved current since

$$\overset{(2)}{J}_{\mu\nu}(x) \approx T_{\mu\nu}(x) + \frac{1}{4} \left(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right) |\phi(x)|^2 \,. \qquad (3.8)$$

3.2 Noether interactions

The conserved currents $J^{(r)}$ of eq. (3.6) allow to define Noether interactions between the scalar ϕ and gauge fields $h^{(r)}$ as in eq. (2.10) by

$$S_1[\phi, h] = -\sum_{r=0}^{\infty} \frac{1}{r!} \int d^n x \, \overset{(r)}{h}_{\mu_1...\mu_r}(x) \, \overset{(r)}{J}{}^{\mu_1...\mu_r}(x) \qquad (3.9)$$

$$= -\sum_{r=0}^{\infty} \left(\frac{i}{2}\right)^r \sum_{s=0}^{r} \frac{(-1)^s}{s! \, (r-s)!} \int d^n x \, \overset{(r)}{h}_{\mu_1...\mu_r}(x) \, \partial^{\mu_1} \ldots \partial^{\mu_s} \phi(x) \, \partial^{\mu_{s+1}} \ldots \partial^{\mu_r} \phi^*(x).$$

Similar Noether interactions with scalar field conserved currents were elaborated in [13, 15, 16]. Actually, the above cubic interaction is precisely of the form mentioned in [17], as can be seen from eq. (3.6). The sum of terms in the cubic interaction (3.9) can be expressed in a concise way exhibiting in a manifest way its symmetries. In order to do so, we first introduce the generating function of gauge fields:[4]

$$h(x, v) = \sum_{r=0}^{\infty} \frac{1}{r!} \, \overset{(r)}{h}_{\mu_1...\mu_r} \, v^{\mu_1} \ldots v^{\mu_r}, \qquad (3.10)$$

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[4]From (3.1) and (3.4), it is clear that the generating function $J(x, u)$ has mass dimension $n-2$. Thus (2.8) shows that $h(x, v)$ has mass dimension 2. Since v^μ has the dimension of a mass, the tensor gauge field $h^{(r)}$ of rank r has mass dimension $2 - r$.

so that the Noether interaction (3.9) can be expressed as $\ll J\|h\gg$ or from (2.8) as

$$\int d^n x \ \exp\left(\frac{\partial}{\partial u_\mu}\,\frac{\partial}{\partial v^\mu}\right) \breve{\rho}(x,-iu)\,h(x,v)\,\Big|_{u=v=0}.\tag{3.11}$$

Next, we notice that for any function $f(p)$ and $g(p)$:

$$\exp\left(\frac{\partial}{\partial u_\mu}\,\frac{\partial}{\partial v^\mu}\right)\,\breve{f}(-iu)\,g(v)\,\Big|_{u=v=0} = \int \frac{d^n p}{(2\pi)^n}\ f(p)\,g(p)\,,\tag{3.12}$$

where $\breve{f}(q)$ is the Fourier transform of $f(p)$:

$$\breve{f}(q^\nu) := \int \frac{d^n p}{(2\pi)^n}\ f(x^\mu, p_\nu)\ e^{i\,u^\mu p_\mu}\,.$$

This allows us to to express (3.11) as

$$S_1[\phi, h] = -\frac{1}{(2\pi)^n}\int d^n x\, d^n p\ h(x,p)\ \rho(x,p)\,.\tag{3.13}$$

where $\rho(x,p)$ is the inverse Fourier transform of $\breve{\rho}(x,q)$ over the auxiliary variables q. The form (3.13) of the cubic interaction will be an essentiel ingredient in exhibiting all the symmetries of the cubic action in the next subsection.

If we rewrite the expression (3.13) in momentum space, we can get an even more compact form. By noticing that the Fourier transform of $\rho(x,p)$ over spacetime variables x reads

$$\begin{aligned}
\tilde{\rho}(k,p) &= \int d^n x\, d^n q\ e^{-i\,(k\cdot x + p\cdot q)}\ \phi^*(x-q/2)\,\phi(x+q/2)\\
&= \tilde{\phi}^*(p-k/2)\ \tilde{\phi}(p+k/2)\,,
\end{aligned}\tag{3.14}$$

the Noether interaction can be written in a very simple form in terms of $\tilde{\phi}$:

$$S_1[\phi, h] = -\int \frac{d^n \ell}{(2\pi)^n}\,\frac{d^n k}{(2\pi)^n}\ \tilde{\phi}^*(\ell)\ \tilde{h}\left(\ell - k, \frac{k+\ell}{2}\right)\ \tilde{\phi}(k)\,.\tag{3.15}$$

3.3 Weyl formulation

Using the bra-ket notation for the scalar field $\phi(x) = \langle\, x\mid\phi\,\rangle$, the current generating function $\breve{\rho}(x,q)$ can be written as $\langle\, x+q/2\mid\phi\,\rangle\langle\,\phi\mid x-q/2\,\rangle$. A very important observation is that, as explained in appendix A, this is the Fourier transform over momentum space of the Wigner function $\rho(x,p)$ associated to the operator $\mid\phi\,\rangle\langle\,\phi\mid$:

$$\rho(x,p) = \int d^n q\ e^{-i\,p\cdot q}\ \langle\, x+q/2\mid\phi\,\rangle\langle\,\phi\mid x-q/2\,\rangle\,,\tag{3.16}$$

Thus, the expression of the Noether coupling (3.11) can now be simplified using the Weyl correspondence to

$$S_1[\phi, h] = -\langle\,\phi\mid\hat{H}\mid\phi\,\rangle\,,\tag{3.17}$$

where $\hat{H} := \mathcal{W}[h]$ is the image of the generating function $h(x,p)$ under the Weyl map $\mathcal{W}$ introduced in (A.1). Consequently, the Noether interaction (2.8) defined by the generating functions (3.4) and (3.10) can be written as the "mean value" over the state $|\phi\rangle$ of the operator $\hat{H}$. The expression (3.15) could also have been obtained by inserting the completeness relations $\int d^n k/(2\pi)^n \, |k\rangle\langle k| = \hat{1}$ between each state in (3.17) and apply the identity (A.10). A cubic interaction with scalar matter was written in this form by Segal in the somewhat different context of conformal higher-spin gauge theory [7].

By making use of the "anticommutator ordering" prescription for the Weyl map, as explained in the appendix A, one finds that the operator $\hat{H}$ starts at lower spin as

$$\hat{H} = \overset{(0)}{h}(\hat{X}) + \frac{1}{2} \left(\hat{P}^\mu \overset{(1)}{h}_\mu(\hat{X}) + \overset{(1)}{h}_\mu(\hat{X}) \hat{P}^\mu \right)$$
$$+ \frac{1}{8} \left(\hat{P}^\mu \hat{P}^\nu \overset{(2)}{h}_{\mu\nu}(\hat{X}) + 2 \hat{P}^\mu \overset{(2)}{h}_{\mu\nu}(\hat{X}) \hat{P}^\nu + \overset{(2)}{h}_{\mu\nu}(\hat{X}) \hat{P}^\mu \hat{P}^\nu \right) + \dots \quad (3.18)$$

As one can check, the Noether coupling with the vector gauge field $h^{(1)}_\mu$ is the usual electromagnetic coupling. The Noether coupling with the symmetric tensor gauge field $h^{(2)}_{\mu\nu}$ corresponds to the "minimal" coupling between a spin-two gauge field and a scalar density ϕ of weight one-half (minimal in the sense that there is no term containing the trace $\eta^{\mu\nu} h^{(2)}_{\mu\nu}$ corresponding to the linearised volume element in the interaction). This means that $|\phi|^2$ must be a density of weight one. As can be checked directly from (3.8), if the action (2.11) includes the rank-two conserved current (3.7) only, then it reads

$$S[\phi, h] = - \int d^n x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \Phi^*(x) \partial_\nu \Phi(x) + \left(m^2 - \frac{R}{8} \right) |\Phi(x)|^2 \right] + \mathcal{O}(h^2), \quad (3.19)$$

in terms of the scalar $\Phi := (-g)^{-\frac{1}{4}} \phi$, the metric $g_{\mu\nu} := \eta_{\mu\nu} + h^{(2)}_{\mu\nu} + \mathcal{O}(h^2)$ and the scalar curvature R.

It is worth emphasising that the cubic interaction $S_1[\phi, h]$ contains r derivatives and grows like the power $r - 3 + n/2$ of the energy scale by naive dimensional analysis, so if it involves a tensor field of rank $r > 3 - n/2$ then it is not (power-counting) renormalisable. Notice also that for a real scalar field, the interactions occur with tensor gauge fields of even rank only.

3.4 Weyl algebra as a non-Abelian gauge symmetry

Using the braket notation of scalar field where $\partial_\mu \phi(x) = i \langle x| \hat{P}_\mu |\phi\rangle$ and the completeness relation $\hat{1} = \int d^n x \, |x\rangle\langle x|$, the Klein-Gordon action (3.1) can be rewritten as

$$S_0[\phi] = - \langle \phi| \hat{P}^2 + m^2 |\phi\rangle, \quad (3.20)$$

which is (minus) the mean value over the state $|\phi\rangle$ of the Hamiltonian (constraint) $\hat{P}^2 + m^2$. The quadratic and cubic functionals (3.20) and (3.17) are such that the would-be action (2.11) at all orders in the gauge fields starts as

$$S[\phi, h] = - \langle \phi| \hat{G} |\phi\rangle + \mathcal{O}(\phi^3, h^2), \quad (3.21)$$

where the operator

$$\hat{G} := \hat{P}^2 + m^2 + \hat{H} \,, \tag{3.22}$$

should be interpreted in terms of its Weyl symbol

$$g(x,p) := p^2 + m^2 + h(x,p) \,,$$

as the generating function of the various gauge fields around the Minkowski metric as background.

The linearised gauge transformation (2.4) of the Weyl symbol $h(x,p)$ can be written as the Poisson bracket between the function $\varepsilon(x,p)$ and the Weyl symbol of $p^2 + m^2$ of a free relativistic particle,

$$\left(p^\mu \frac{\partial}{\partial x^\mu} \right) \varepsilon(x,p) = \frac{1}{2} \left\{ \varepsilon(x,p), \ p^2 + m^2 \right\}_{\mathbf{P.B.}} = -\frac{i}{2} \left[\varepsilon(x,p) \overset{\star}{,} p^2 + m^2 \right] , \quad (3.23)$$

where $\{\cdot,\cdot\}_{\mathbf{P.B.}}$ is the Poisson bracket and $[\cdot \overset{\star}{,} \cdot]$ is the commutator with respect to the Moyal product. The image of the above formula under the Weyl map leads to

$$\delta_{\hat{E}} \hat{H} = -\frac{i}{2} \left[\hat{E} , \ \hat{P}^2 + m^2 \right] + \mathcal{O}(\hat{H}) \,, \tag{3.24}$$

where $\hat{E}$ is the image of $\varepsilon(x,p)$ under the Weyl map. The variation of the scalar field ϕ which guarantees the gauge invariance, at lowest order in h, of the action (3.21) is

$$\delta_{\hat{E}} |\phi\rangle = -\frac{i}{2} \hat{E} |\phi\rangle \,, \tag{3.25}$$

as can be checked directly. At lower orders in the derivative, the explicit form of the operator $\hat{E}$ in terms of its Weyl symbol $\varepsilon(x,p)$

$$\hat{E} = \overset{(0)}{\varepsilon}(\hat{X}) + \underbrace{\frac{1}{2} \left(\hat{P}^\mu \overset{(1)}{\varepsilon}_\mu(\hat{X}) + \overset{(1)}{\varepsilon}_\mu(\hat{X}) \hat{P}^\mu \right)}_{= -i\left(\overset{(1)}{\varepsilon}_\mu(\hat{X}) \partial^\mu + \frac{1}{2} \partial^\mu \overset{(1)}{\varepsilon}_\mu(\hat{X}) \right)} + \dots \tag{3.26}$$

confirms that following (3.25) the matter field ϕ transforms as a scalar density of weight one-half under the (linearised) diffeomorphisms. The set of all such transformations (3.25) closes under the commutator and is isomorphic to the Lie algebra of Hermitian operators, i.e. the Lie algebra of quantum observables, corresponding to the Lie group of unitary operators. If one truncates the tower of gauge fields to the lower-spin sector, then there are no further terms represented by dots in (3.26), and the Lie algebra of symmetries one is left with is the semidirect sum of the local $\mathfrak{u}(1)$ algebra and the algebra of vector fields on $\mathbb{R}^n$, corresponding to the semidirect product of the local $U(1)$ group and the group of diffeomeorphisms. The form of (3.21) suggests the following finite gauge transformation

$$|\phi\rangle \longrightarrow \hat{U} |\phi\rangle \,, \qquad \hat{G} \longrightarrow \hat{U} \hat{G} \hat{U}^{-1} \,, \tag{3.27}$$

with $\hat{U} := \exp(-i\,\hat{E}/2)$, because, *at lowest order* in $\hat{H}$, it reproduces the infinitesimal transformations (3.24)–(3.25) and leaves invariant the quadratic form $\langle\phi|\hat{G}|\phi\rangle$. The

scalar and gauge fields respectively transform in the fundamental and adjoint representation of the group of unitary operators. Notice that as long as higher-derivative transformations are allowed then the infinite tower of higher-spin fields should be included for consistency of the gauge transformations (3.27) beyond the lowest order. The infinitesimal version of (3.27) written in terms of the Weyl symbols leads to the following completion of (2.4)

$$\delta_\varepsilon \, h(x,p) = -\frac{i}{2} \left[\, \varepsilon(x,p) \stackrel{\star}{,} p^2 + m^2 + h(x,p) \,\right] \tag{3.28}$$
$$= \left(\eta^{\mu\nu} \, p_\mu \, \frac{\overrightarrow{\partial}}{\partial x^\nu} - h(x,p) \, \sin\left[\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu} \, \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu} \, \frac{\overrightarrow{\partial}}{\partial x^\mu} \right)\right] \right) \varepsilon(x,p),$$

where we made use of (A.17) and (3.23). Such a deformation of the higher-spin gauge transformations was already advocated in [7, 18, 19].[5] Notice that, in general, the Moyal bracket contains a non-vanishing contribution at $p_\mu = 0$ which corresponds in (3.28) to a gauge transformation of a tensor field of rank $r = 0$. Hence, it might be necessary for the consistency of the non-Abelian gauge transformations (3.28) to include a scalar field $h^{(0)}$ in the tower of gauge fields.[6]

The Weyl symbol $\overline{\varepsilon}(x,p)$ of an operator $\hat{\overline{E}}$ commuting with $\hat{P}^2 + m^2$ is a generating function of Killing fields, as can be easily seen from (3.23). This is in agreement with the facts that if $[\,\hat{\overline{E}},\,\hat{P}^2 + m^2\,] = 0$ then the corresponding transformation (3.27),

$$|\,\phi\,\rangle \;\longrightarrow\; \exp(-i\,\hat{\overline{E}}\,)\,|\,\phi\,\rangle, \tag{3.29}$$

is obviously a symmetry of the Klein-Gordon action (3.1). It is very tempting to conjecture that the full action (3.21) should be interpreted as arising from the gauging of the rigid symmetries (3.29) of the free scalar field, which generalise the U(1) and Poincaré symmetries, so the local symmetries (3.27) generalise the local U(1) and diffeomorphisms. The rigid higher-derivative symmetries which are generated by a function $\overline{\varepsilon}(p)$ independent of the position and which thereby generalise the phase shifts and translations were introduced in [13] and further developed in [14]. The corresponding infinitesimal symmetries are the most general rigid linear symmetry transformations of a free scalar field which are independent of the coordinates and compatible with locality (in the sense that the order of the differential operators is finite). The group of unitary operators was already advertised in [16, 18] as the symmetry group arising from the gauging of these rigid higher-derivative symmetries.

Notice that the conserved currents (3.6) are indeed equivalent to the Noether currents for the latter symmetries, as follows from Noether's first theorem or as can be checked by direct computation. This correspondence also implies that the Noether interaction considered here is the most general one (up to equivalence) between one gauge field and two free scalars that induces a gauge transformation of the scalar field and is compatible with locality and Poincaré symmetry. In the case of a real scalar field, the Lie algebra

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[5]This deformation was already implicit in [20] in the sense that (3.28) should arise after the elimination of the auxiliary variables y.

[6]A scalar field is also necessary for consistency of Vasilev's unfolded equations. It should be stressed that the transformation (3.28) of the gauge scalar field $h^{(0)}$ is distinct from the transformation (3.25) of the matter scalar field ϕ.

and group of gauge symmetries would have to be replaced by, respectively, the algebra of symmetric operators and the group of orthogonal operators. The former construction goes along the same line for a scalar field taking values in an internal finite-dimensional space, i.e. for a multiplet of scalar fields.
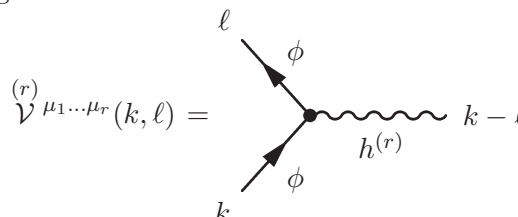
4 Tree-level higher-spin exchange amplitudes

4.1 Feynman rules

Vertex. The cubic vertex between two scalar fields ϕ and a gauge field $h^{(r)}$ takes a simple form in momentum space in terms of the Fourier transforms of fields, $\tilde{\phi}$. Indeed, from eq. (3.15), the Noether interaction between ϕ and $h^{(r)}$ is given by

$$S_1 \left[\phi, h^{(r)} \right] = - \int \frac{d^n \ell}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \; \tilde{\phi}^*(\ell) \, \tilde{\phi}(k) \; \overset{(r)}{\tilde{h}}_{\mu_1 \ldots \mu_r}(\ell - k) \times$$
$$\times \frac{1}{r!} \left(\frac{k^{\mu_1} + \ell^{\mu_1}}{2} \right) \ldots \left(\frac{k^{\mu_r} + \ell^{\mu_r}}{2} \right) . \tag{4.1}$$

The corresponding cubic vertices is

$$\overset{(r) \, \mu_1 \ldots \mu_r}{\mathcal{V}}(k, \ell) = \quad$$

$$= - \frac{1}{r!} \left(\frac{k^{\mu_1} + \ell^{\mu_1}}{2} \right) \ldots \left(\frac{k^{\mu_r} + \ell^{\mu_r}}{2} \right) . \tag{4.2}$$

If the scalar field is real then one can insert the relation $\tilde{\phi}^*(-k) = \phi(k)$ in (4.1) and recover the fact that cubic vertices for odd r are absent in such case.

Propagators. The propagator with respect to the scalar field ϕ is easily determined from the kinetic term in (3.20) and is given by

$$\mathcal{D}(p) = \quad$$

$$= \frac{1}{p^2 + m^2} . \tag{4.3}$$

The current-current interaction which determines the propagator $\mathcal{P}^{(r)}/p^2$ for spin r exchange was determined in [9], and the amplitudes were shown to propagate the correct numbers of on-shell degrees of freedom, exactly like in Fronsdal's formulation, even though the currents involved were not doubly traceless. The contraction of the propagator residue $\mathcal{P}^{(r)}$ with two conserved currents $J_1^{(r)}$ and $J_2^{(r)}$ is given by

$$\left\langle \overset{(r)}{J}_1 , \overset{(r)}{\mathcal{P}} \overset{(r)}{J}_2 \right\rangle = \sum_{m=0}^{\lfloor r/2 \rfloor} \frac{1}{2^{2m} \, m! \, (3 - \frac{n}{2} - r)_m} \left\langle \overset{(r)}{J}_1{}^{[m]} , \overset{(r)}{J}_2{}^{[m]} \right\rangle , \tag{4.4}$$

where $J^{[m]}$ denotes the m-th trace of the external current, $(a)_m$ denotes the m-th Pochhammer symbol of a: $(a)_m = \Gamma(a+m)/\Gamma(a)$.

This amplitude corresponds to a kinetic term for the spin r fields which is canonically normalised that is of the form $S_{\text{kin}}[h^{(r)}] = \frac{1}{2} < h^{(r)} | \Box h^{(r)} > + \dots$ However, our gauge fields $h^{(r)}$ are not canonically normalised and the kinetic terms compatible, to lowest order, with our symmetries have the form

$$S_{\text{kin}}[h^{(r)}] = \frac{\lambda^{6-n-2r}}{2 a_r} \left\langle \overset{(r)}{h} | \Box \overset{(r)}{h} \right\rangle + \dots \tag{4.5}$$

with λ a length parameter and a_r real strictly positive dimensionless parameters. Thus, the propagator with respect to our $h^{(r)}$ is given, in a gauge à la Feynman and de Donder, by

$$\overset{(r)}{\mathcal{D}}_{\mu_1 \dots \mu_r \,|\, \nu_1 \dots \nu_r}(k) = \mu_1 \dots \mu_r \,\,\, \sim\!\!\sim\!\!\sim\!\!\sim\!\!\sim\!\!\sim \,\,\, \nu_1 \dots \nu_r$$

$$= \frac{a_r \,\lambda^{n-6+2r}}{k^2} \overset{(r)}{\mathcal{P}}_{\mu_1 \dots \mu_r \,|\, \nu_1 \dots \nu_r}. \tag{4.6}$$

4.2 Tree-level amplitude

We consider the following diagram where two scalar particles of same charge exchange one gauge particle of rank r in t-channel.
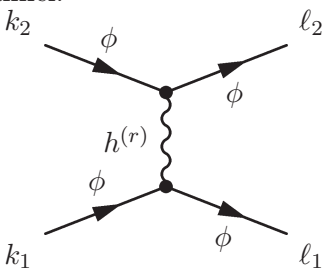
Since vertices $\mathcal{V}^{(r)}$ are conserved, the corresponding amplitude is given by

$$\overset{(r)}{A}(\phi(k_1)\phi(k_2) \rightarrow \phi(\ell_1)\phi(\ell_2)) = \left\langle \overset{(r)}{\mathcal{V}}(k_1, \ell_1), \overset{(r)}{\mathcal{D}}(k_1 - \ell_1) \overset{(r)}{\mathcal{V}}(k_2, \ell_2) \right\rangle, \tag{4.7}$$

where the contraction notation (2.7) was used and the momentum conservation $k_1 + k_2 = \ell_1 + \ell_2$ is assumed. We recall that the propagator $\mathcal{D}$ is given in (4.6) and (4.4). Using eq. (4.2), the contraction between two m-th traces of vertex is given by

$$\left\langle \overset{(r)}{\mathcal{V}}{}^{[m]}(k_1, \ell_1), \overset{(r)}{\mathcal{V}}{}^{[m]}(k_2, \ell_2) \right\rangle =$$

$$= \frac{1}{(r-2m)!} \left[\frac{k_1 + \ell_1}{2} \cdot \frac{k_2 + \ell_2}{2} \right]^{r-2m} \left[\left(\frac{k_1 + \ell_1}{2} \right)^2 \left(\frac{k_2 + \ell_2}{2} \right)^2 \right]^m. \tag{4.8}$$

By making use of the above result and the Mandelstam variables s, t and u (see appendix B for more details):

$$(k_1 + \ell_1) \cdot (k_2 + \ell_2) = -(s - u),$$
$$(k_1 + \ell_1)^2 = (k_2 + \ell_2)^2 = -(s + u),$$
$$(k_1 - \ell_1)^2 = (k_2 - \ell_2)^2 = -t, \tag{4.9}$$

the amplitude can be written as[7]

$$\overset{(r)}{A}(s,t,u) = -\frac{\lambda^{n-6}}{t}\, a_r \left(-\frac{\lambda^2}{4}\right)^r \sum_{m=0}^{\lfloor r/2 \rfloor} \frac{(s-u)^{r-2m}(s+u)^{2m}}{2^{2m}\, m!\,(r-2m)!\,(3-\frac{n}{2}-r)_m}\,. \tag{4.10}$$

In four-dimensional spacetime ($n=4$) and for $r \geqslant 1$, the sum (4.10) can be expressed in terms of Chebyshev polynomials of the first kind (C.2) as

$$\overset{(r)}{A}(s,t,u) = -\frac{\lambda^{-2}}{t}\, a_r \left(-\frac{\lambda^2}{8}(s+u)\right)^r \frac{2}{r!}\, T_r\left(\frac{s-u}{s+u}\right)\,. \tag{4.11}$$

In higher dimensions ($n \geqslant 5$), the sum (4.10) can be expressed in terms of Gegenbauer polynomials (C.3) as

$$\overset{(r)}{A}(s,t,u) = -\frac{\lambda^{n-6}}{t}\, a_r \left(-\frac{\lambda^2}{8}(s+u)\right)^r \frac{1}{(\frac{n}{2}-2)_r}\, C_r^{\frac{n}{2}-2}\left(\frac{s-u}{s+u}\right)\,. \tag{4.12}$$

Notice that in $n=5$ dimensions, the Gegenbauer polynomial in (4.12) essentially becomes a Legendre polynomial. These amplitudes have a pole when t is equal to the squared mass of an exchanged particle. Thus for massless mediators t must be different from zero, i.e. the scattering angle $\theta \neq 0$ modulo π.[8]

For bosons, the total amplitude for the scattering process $\phi(k_1)\,\phi(k_2) \to \phi(\ell_1)\,\phi(\ell_2)$ contains the sum of the t and u channel amplitude:

$$= \overset{(r)}{A}(s,t,u) \qquad + \qquad \overset{(r)}{A}(s,u,t)\,. \tag{4.14}$$

The diagrams for the scattering $\phi(k_1)\,\bar\phi(k_2) \to \phi(\ell_1)\,\bar\phi(\ell_2)$ can be obtained from $A^{(r)}$ by a *crossing* symmetry:

$$\tag{4.15}$$

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[7]The case $r=0$ corresponds to the exchange of a scalar "gauge" field $h^{(0)}$ and so is slightly less natural from a physical perspective than cases $r \geqslant 1$.

[8]The scattering angle θ in the center-of-mass system is determined by

$$\sin^2(\theta/2) = -t/(s-4\,m^2)\,, \qquad \cos^2(\theta/2) = -u/(s-4\,m^2)\,. \tag{4.13}$$

Since $s \geqslant 4m^2$, one should have $t \leqslant 0$ and $u \leqslant 0$. See appendix B for more details.

The parity properties of Gegenbauer and Chebyshev polynomials are such that

$$\overset{(r)}{A}(u,t,s) = (-1)^r \overset{(r)}{A}(s,t,u) \,, \tag{4.16}$$

which is consistent with crossing "symmetry." For instance, if the scalar field is real then the amplitude is a symmetric function of s and u.

If λ is thought as Planck's length and m as, say, the proton mass, then $\lambda m \approx 10^{-19} \ll 1$. The high-energy regime must now be understood as $s \gg \lambda^{-2} \gg m^2$. In the Regge limit, the t-channel tree-level amplitudes behave as

$$\overset{(r)}{A}(s,t,u) \sim - \frac{\lambda^{n-6}}{t} \frac{a_r}{r!} \left(-\frac{\lambda^2}{2} s \right)^r \,,$$

and for fixed scattering angle θ in $n = 4$ as

$$\overset{(r)}{A}(s,t,u) \sim - \frac{1}{4} \frac{a_r}{r!} \left(-\frac{\lambda^2}{8} \sin^2(\theta/2) s \right)^{r-1} T_r \left(\frac{1 + \cos^2(\theta/2)}{\sin^2(\theta/2)} \right) \,.$$

As one can see, in the latter limit each amplitude grows as the $(r-1)$-th power of the large s, so it goes to a constant when $r = 1$ and it diverges for spin $r \geqslant 2$. This is another signal of the well-known fact that the corresponding interactions are or not (power-counting) renormalisable.

5 Summation of tree amplitudes and high-energy behaviour

In the present section, the main focus is on spacetime dimension $n = 4$ for obvious physical reasons (and because the case $n \geqslant 5$ goes exactly along the same lines). For the process $\phi\phi \to \phi\phi$, the sum of the t-channel tree-level amplitudes including all exchanged particles is

$$A(s,t,u) = \sum_{r \geqslant 0} \overset{(r)}{A}(s,t,u) = -\frac{\lambda^{-2}}{t} \left[a_0 + \sum_{r \geqslant 1} a_r \left(-\frac{\lambda^2}{8}(s+u) \right)^r \frac{2}{r!} T_r \left(\frac{s-u}{s+u} \right) \right] \,. \tag{5.1}$$

Let us denote by $a(z)$ the generating function of the coefficients $a_r (\geqslant 0)$, in the sense that

$$a(z) = \sum_{r \geqslant 0} \frac{a_r}{r!} z^r \,. \tag{5.2}$$

Using the identity (C.1), the sum (5.1) over r can be explicitly performed and gives

$$A(s,t,u) = -\frac{\lambda^{-2}}{t} \left[a \left(-\frac{\lambda^2}{8} \left(\sqrt{s} + \sqrt{-u} \right)^2 \right) + a \left(-\frac{\lambda^2}{8} \left(\sqrt{s} - \sqrt{-u} \right)^2 \right) - a_0 \right] \,. \tag{5.3}$$

In the high-energy regimes $s \gg \lambda^{-2} \gg m^2$, the t-channel tree amplitude behaves in the Regge limit as

$$-\frac{\lambda^{-2}}{t} a \left(-\frac{\lambda^2}{2} s \right) \,, \tag{5.4}$$

and in the fixed scattering angle limit as

$$\frac{\lambda^{-2}}{\sin^2(\theta/2) s} \left[a \left(-\frac{\lambda^2}{8} [1 - \cos(\theta/2)]^2 s \right) + a \left(-\frac{\lambda^2}{8} [1 + \cos(\theta/2)]^2 s \right) - a_0 \right] \,, \tag{5.5}$$

which formally reproduces the behaviour (5.4) in the limit $\theta \to 0$ with fixed (but large) s.

5.1 Simplest examples

We first consider the simplest choice of coefficients: $a_r = 1$ for all $r \geqslant 0$. Hence $a(z) = e^z$ so that the t-channel amplitude is equal to

$$A(s,t,u) = -\frac{\lambda^{-2}}{t} \left[2 \exp\left(-\frac{\lambda^2}{8}(s-u) \right) \cosh\left(\frac{\lambda^2}{4} \sqrt{-su} \right) - 1 \right], \qquad (5.6)$$

and decreases exponentially in the Regge limit,

$$A(s,t,u) \sim -\frac{\lambda^{-2}}{t} \exp\left(-\frac{\lambda^2}{2} s \right)$$

in agreement with (5.4). Next, in order to cancel the constant contribution in the brackets of (5.6) we consider another choice of coefficients $a_0 = 2$ and $a_r = 1$ for all $r \geqslant 1$. Hence $a(z) = e^z + 1$ and the t-channel amplitude is equal to

$$A(s,t,u) = -\frac{2\lambda^{-2}}{t} \exp\left(-\frac{\lambda^2}{8}(s-u) \right) \cosh\left(\frac{\lambda^2}{4} \sqrt{-su} \right),$$

and falls-off exponentially for large s but fixed scattering angle $\theta \neq 0$

$$A(s,t,u) \sim \frac{\lambda^{-2}}{\sin^2(\theta/2)\, s} \exp\left(-\frac{\lambda^2}{8} \left[1 - \cos(\theta/2) \right]^2 s \right)$$

as can be checked directly or from (5.5). However, the t-channel tree-level scattering amplitude of the process $\phi\bar{\phi} \to \phi\bar{\phi}$ grows exponentially.

5.2 General discussion

Let $a(z)$ be the real function defined by the power series (5.2) with non-negative coefficients $a_r \geqslant 0$. Let us assume that the function is holomorphic on the complex plane except a set of isolated poles (i.e. it is meromorphic) which does not contain the origin. More concretely, the function $a(z)$ is analytic inside the disk of convergence of the power series $\sum_{r \geqslant 0} \frac{a_r}{r!} z^r$ around the origin $z = 0$ and it is defined outside the radius of convergence by analytic continuation.

The poles of the corresponding t-channel tree-level amplitude for the exchange of an infinite tower of tensor gauge fields between two scalar particles might be interpreted, effectively, as the exchange of some massive particles. This amplitude goes to zero in the Regge limit if and only if $z = -\infty$ is a zero of $a(z)$, as can be seen from (5.4). Moreover, at any fixed scattering angle $\theta \neq 0$ (modulo π), the high-energy limit of the t-channel tree-level amplitude goes to zero if $z = 0$ is another zero of $a(z)$, as follows from (5.5). The crossing transformation $s \leftrightarrow u$ of the amplitude (5.3) is equivalent to the exchange $a(z) \leftrightarrow a(-z)$. Therefore, the t-channel tree-level amplitude for the scattering process $\phi\bar{\phi} \to \phi\bar{\phi}$ also goes to zero in the ultraviolet if the analytic function $a(z)$ has another zero at $z = +\infty$. Unfortunately, this is not possible if the power series defining $a(z) \neq 0$ around zero is convergent on the whole positive axis because all coefficients $a_r \geqslant 0$ of the power series of $a(z)$ are non-negative. An interesting possibility is therefore when the function $a(z)$

has a finite radius of convergence around the origin. Outside the disk of convergence, the function may be analytically continued and it is this analytic continuation which determines the high energy behaviour. A simple example is given by $a_r = r!$ in which case the analytic continuation is given by $a(z) = (1-z)^{-1}$ which vanishes for any large argument $z = \pm\infty$.[9]

Consequently, the total scattering amplitude may be extremely soft in the ultraviolet regime, though any individual exchange amplitude grows quickly (for spin $r \geqslant 2$). Such asymptotic behaviours may be qualitatively understood as follows: The t-channel scattering amplitude of the process $\phi\phi \to \phi\phi$ corresponding to an exchange of a tensor field of rank r behaves polynomially in the Regge limit like $(-s)^r/t$, which is more and more divergent for larger rank r. However, very precisely along the lines of [17], one may observe that the asymptotic behaviour of, say, the power series $\sum_{r\geqslant 0}(-s)^r/r! = e^{-s}$ is much smaller when $s \to +\infty$ than any individual term. Such a property arises naturally for current-current interactions between two scalar particles ϕ (or two scalar antiparticles $\bar{\phi}$) because the series is alternating with the rank r (remember that the coefficients are non-negative $a_r \geqslant 0$). Heuristically, some compensations are possible between the exchanges of even-spin (attractive) and odd-spin (repulsive) gauge tensors. Naively, this mechanism seems impossible between a scalar particle ϕ and its antiparticle $\bar{\phi}$ or if the scalar field is real ($\phi = \bar{\phi}$) because, intuitively, the former interactions are always attractive. More precisely, the t-channel scattering amplitude of the $\phi\bar{\phi} \to \phi\bar{\phi}$ corresponding to an exchange of a tensor field of rank r behaves polynomially in the Regge limit like s^r/t. Actually, the asymptotic behaviour is a subtle issue because the non-compensation argument works only inside the disk of convergence of the power series defining the amplitude. Indeed, for instance the power series we already mentioned $\sum_{r\geqslant 0} s^r = (1-s)^{-1}$ is not alternating but its analytic continuation $(1-s)^{-1}$ goes to zero when $s \to \infty$.

5.3   Softness and finiteness

The softness of tree-level scattering amplitudes in the high-energy regime is a strong indication in favour of ultraviolet finiteness. For instance, various loop diagrams can be built out of the (off-shell) diagram of the previous section. As an illustration, one may consider the following one-loop contribution to the scalar propagator (encoding its self-energy)

$$ k \longrightarrow\!\!\!\overbrace{}\!\!\!\longrightarrow k \quad \propto \quad \int d^4 p \, \frac{A\left(\phi(k)\phi(p) \to \phi(-p)\phi(k)\right)}{p^2 + m^2}, $$

---

[9]This example exhibits a general feature: If the real function $a(z)$ defined by the power series (5.2) with non-negative coefficients $a_r \geqslant 0$ is meromorphic and has a finite radius of convergence $R > 0$ then $z = R$ is a pole of $a(z)$ on the positive axis. The idea of the proof is as follows: The modulus of the function a at any given point z_0 inside the disk of convergence satisfies the inequality $|a(z_0)| \leqslant \sum_r \frac{|a_r|}{r!} |z_0|^r = \sum_r \frac{a_r}{r!} |z_0|^r = a(|z_0|)$, because the coefficients a_r are non-negative. The function $a(z)$ is meromorphic and its power series around the origin has a finite radius of convergence $R > 0$, thus it must have a pole z_0 on the circle of radius R, i.e. $|z_0| = R$ and $|a(z_0)| = \infty$. Therefore $a(|z_0|) = a(R) = \infty$. In other words, $z = R$ is a singularity of $a(z)$ on the positive axis, which can only be a pole because the function is meromorphic.

where the internal curly lines should be understood as the sum over all possible gauge fields and the amplitude A is extended off-shell. This Feynman diagram can be seen to have at most a logarithmic divergence in the UV if $a(z)$ goes to a constant when $z \to \pm\infty$. This is already much better than any individual contribution coming from a finite number of gauge fields of spin $r \neq 0$ in the internal curly line.

Another example is the following box diagram which contributes to the two-scalar scattering process at one-loop.
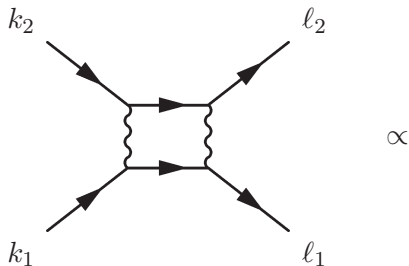
$$\int d^4p \, \frac{A\left(\phi(k_1)\phi(k_2) \to \phi(k_1+p)\phi(k_2-p)\right) \, A\left(\phi(k_1+p)\phi(k_2-p) \to \phi(\ell_1)\phi(\ell_2)\right)}{\left((k_1+p)^2+m^2\right)\left((k_2-p)^2+m^2\right)},$$

This Feynman diagram can be seen to be UV finite if $a(z)$ goes to some constant when $z \to \pm\infty$. Of course, this does not imply that the corresponding total one-loop amplitudes are finite because other diagrams should be taken into account, some of which might include higher-order vertices which are not considered in the present paper. Nevertheless, it is already very suggestive to observe that some Feynman diagrams may be UV finite if all contributions of the whole infinite tower of gauge fields are summed.

6   Non-relativistic interaction potential

Since the higher-spin particles are massless, one may wonder about the macroscopic interactions that they give rise to in $n = 4$ dimensions. In the low-energy (or non-relativistic) regime the Mandelstam variables of the scattering process $\phi\phi \to \phi\phi$ examined in section 4 behave as $s \sim 4m^2$ and $|u| \ll s$, thus the t-channel exchange amplitudes are equal to

$$\overset{(r)}{A}\left(\phi(\vec{k})\,\phi(-\vec{k}) \to \phi(\vec{\ell})\,\phi(-\vec{\ell})\right) \sim -\frac{a_r}{r!}\left(-\frac{(m\,\lambda)^2}{2}\right)^{r-1}\frac{m^2}{(\vec{k}-\vec{\ell})^2}. \qquad (6.1)$$

For the interaction arising from the exchange of a spin-r mediator, the non-relativistic potential between two elementary scalar particles separated by $\vec{x}$ can be deduced from the above amplitude (6.1) via the Born approximation (B.3) and reads

$$\overset{(r)}{V}(\vec{x}) = \frac{a_r}{4\,r!}\left(-\frac{(m\lambda)^2}{2}\right)^{r-1}\frac{1}{4\pi\,|\vec{x}|}, \qquad (6.2)$$

The sign indicates that even (odd) spin massless particles mediate attractive (repulsive) interactions between identical scalar particles (i.e. charges of the same sign). The effective non-relativistic potential including all possible exchanges is the sum

$$V(\vec{x}) := \sum_{r \geqslant 0} \overset{(r)}{V}(\vec{x}) = -\frac{1}{2\,(m\,\lambda)^2}\,a\left(-\frac{(m\,\lambda)^2}{2}\right)\frac{1}{4\pi\,|\vec{x}|}. \qquad (6.3)$$

This expression somehow justifies on physical ground the mathematical assumption that the function $a(z)$ should at least be analytic around zero. Indeed, in such case

$$V(\vec{x}) = V_{\text{lower}}(\vec{x}) + \mathcal{O}\left((m\lambda)^4\right)$$

where $m\lambda \ll 1$ and V_{lower} denotes the part of the effective potential corresponding to exchange of lower $(r \leqslant 2)$ spin particles. In other words, the validity of the Taylor expansion of $a(z)$ around zero agrees with the fact that higher-spin contributions are not observable at energy scales much smaller than Planck's mass. In order to bring another perspective on this point, suppose now that we have two macroscopic bodies respectively made of $N \gg 1$ and $N' \gg 1$ charged scalars (each of mass m). The macroscopic bodies have thus respective masses $M = N\,m$ and $M' = N'\,m$. The resulting macroscopic potential energy of the system for the interaction mediated by a massless spin-r field is then obtained from (6.2) and reads

$$\overset{(r)}{W}(\vec{x}) := N\,N'\,\overset{(r)}{V}(\vec{x}) = -\frac{\lambda^2}{8}\frac{a_r}{r!}\left(-\frac{(m\lambda)^2}{2}\right)^{r-2}\frac{M\,M'}{4\pi\,|\vec{x}|}\,,\tag{6.4}$$

All macroscopic interaction potential can be expressed in terms of the spin-two exchange (gravitational interaction) as

$$\overset{(r)}{W}(\vec{x}) = \frac{2\,a_r}{r!\,a_2}\left(-\frac{(m\lambda)^2}{2}\right)^{r-2}\overset{(2)}{W}(\vec{x})\,,\tag{6.5}$$

which clearly shows that when $m\lambda \ll 1$ (e.g. if we take for m the proton mass and for λ the inverse of the Planck mass) the higher-spin interaction are negligible compared to the gravitational ones. In order for the scalar exchange contribution to be macroscopically invisible, one may assume that $a_0 \ll (m\lambda)^4 \ll 1$. Moreover, if the macroscopic bodies are approximately "neutral" (same number of particles and antiparticles) then the odd-spin interactions are completely negligible. The toy model considered here allows to understand why higher-spin interactions would not be macroscopically observable if they exist.

7 Conclusion and discussion of results

As advocated here, the Noether procedure applied to an infinite tower of (higher-rank) conserved currents associated with (higher-derivative) symmetries of the Klein-Gordon equation is deeply connected with Weyl quantisation and leads to a gauge symmetry group which is (at lowest order) isomorphic to the group of unitary operators on $\mathbb{R}^n$. In this picture, the scalar field transforms in the fundamental while the tower of symmetric tensor gauge fields transforms in the adjoint representation of this group. Apart from technical complications, the straight analogue of this cubic coupling between a tower of (higher-spin) gauge fields and a free scalar field on any Riemannian manifold $\mathcal{M}$ should lead to the group of unitary operators on $\mathcal{M}$. The only difference would be that the Noether procedure could hold for homogeneous manifolds only, in order for conserved currents to exist. Since only the simplest examples of matter (a scalar field) and background (Minkowski spacetime) have been considered here, the natural questions of how to extend the present analysis for spinor fields and/or for constant-curvature spacetimes arise; they are currently under investigation.

The use of symbol calculus also enables to write the cubic vertex in a very compact form which allows an explicit computation of the general four-scalar tree-level amplitude. The coefficients of the exchanges of symmetric tensor gauge fields may be chosen in such a way that this amplitude is extremely soft in the high-energy regime. For instance, the simplest choice of coefficients leads to an exponential fall-off of the $\phi \phi \rightarrow \phi \phi$ high-energy tree amplitude which is very reminiscent of the behaviour of the ultraviolet fixed-angle Veneziano/Virasoro four-tachyon amplitudes in open/closed string theory. This suggestive property pleads in favour of the standard lore on higher-spin symmetries as the deep origin of ultraviolet softness (and thereby maybe of perturbative finiteness) in string theory. Further evidence in this direction would be provided by fixing the various coefficients from some consistency requirement on the non-Abelian transformations in the gauge field sector.

At first sight, these non-trivial scattering amplitudes and long-range interactions seem in contradiction with the various S-matrix no-go theorems on the interactions between matter and massless higher-spin particles [3, 4]. The main point is that the elastic scattering of matter particles is constrained to be trivial by higher-order conservation laws on products of momenta, as in the case of free or even integrable field theories. For instance, the conservation laws $\sum_i k_i^{\mu_1} \ldots k_i^{\mu_r} = \sum_i \ell_i^{\mu_1} \ldots \ell_i^{\mu_r}$ of order $r > 1$ imply that the outgoing momenta can only be a permutation of the incoming ones. On the one-hand, the low-energy Weinberg theorem [3] states that Lorentz invariance and the absence of unphysical degrees of freedom from the amplitude of the emission of an external soft massless particle of spin r imposes a conservation law of order r. On the other hand, the conservation of higher-spin charges is associated with higher-order conservation laws, as in the Coleman-Mandula theorem [4]. As a corollary, asymptotic higher-spin massless particles or conserved charges imply the triviality of the S-matrix. Like all theorems, the weakness of a no-go theorem relies in its assumptions. In the present case, the fact that the scattering amplitudes of two scalars with some higher-spin field exchanged are non-trivial could have several explanations, among which:

- Asymptotic states of massless higher-spin particles may not exist in the complete theory, similarly to coloured states in QCD.

- It is necessary to fix the gauge in order to define the propagators for massless higher-spin fields, thus it is not obvious that their gauge symmetries automatically imply the existence of non-vanishing higher-spin conserved charges.

- The cubic vertex has been shown to be consistent at lowest order only, while the interactions might become inconsistent at higher-orders.

- There is no genuine S-matrix in (Anti) de Sitter space-time, so even if the cubic vertex is inconsistent in Minkowski space-time, its deformation in curved space-time might be consistent to all orders. In a sense, the AdS/CFT correspondence is the definition of the "S-matrix" in Anti de Sitter space-time [21]. Therefore, an infinite number of asymptotic higher-spin conserved charges means that the holographic dual theory is integrable, but it does not imply that the "scattering" theory in the bulk (defined by the Witten diagrams) is trivial at all. This observation is indeed the very basis of the holographic correspondence in the higher-spin context [22].

- Along these lines, another possibility is that, when $m = 0$, the action $\langle \phi | \hat{G} | \phi \rangle$ could be interpreted as the action for a conformal scalar field ϕ living on the "boundary" of AdS and interacting with higher-spin gauge fields in the "bulk" (see [7] for similar line of reasoning).

To end up, the issue of the trace constraints of Frønsdal [11] on the gauge fields and parameters in higher-spin metric-like theory has not been discussed in the previous sections and deserves some comments. These constraints might have been included by consistently imposing weaker conservation laws on double-traceless currents. This would not modify the current-current interactions because the residue of the propagator is automatically double-traceless, as pointed out in [9]. Nevertheless, it was convenient to remove trace constraints when reflecting on the non-Abelian symmetry group. Anyway, the trace constraints may be removed in the action principle for free higher-spin metric-like fields in several ways (see [23] for some reviews, and [9, 24] for some later developments). As far as the non-Abelian frame-like formulation is concerned, the analogues of Vasiliev's unfolded equations in the unconstrained case [18, 20] are dynamically empty and can somehow be thought [18, 20, 25] of as Fedosov's quantisation [26] of the cotangent bundle along the lines of [27]. But a slight refinement of Vasiliev's unfolded equations [1] has been proposed in [28] and should also be dynamically interesting. The frame-like formalism with weaker trace constraints [29] might also prove to be useful in this respect. Last but not least, the group of gauge symmetries of the metric-like theory arising from unconstrained frame-like theories (by fixing the gauge and solving the torsion constraints) can be shown to be also isomorphic to the group of unitary operators on $\mathbb{R}^n$, at lowest order in the gauge fields and around flat spacetime [19].

Acknowledgments

A   Weyl quantisation

The Weyl formalism [10] offers a classical-like formulation of quantum mechanics using phase space functions as observables and the Wigner function as an analogue of the Liouville density function.

In order to fix the ideas, one may consider the simplest case: the quantum description of a single particle. Classical mechanics is based on the commutative algebra of classical observables (i.e. real functions $f(x^\mu, p_\nu)$ on the phase space $T^* \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^{n*}$) endowed

with the canonical Poisson bracket

$$\{f, g\}_{\mathbf{P.B.}} \;=\; \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} \;-\; \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu} \,.$$

The *Weyl map* $\mathcal{W} : f(x^\mu, p_\nu) \mapsto \hat{F}$ associates to any function f a Weyl(i.e. symmetric)-ordered operator $\hat{F}$ defined by

$$\hat{F} \;=\; \frac{1}{(2\pi\hbar)^n} \int d^n k \, d^n v \; \mathcal{F}(k, v) \, e^{\frac{i}{\hbar} (k_\mu \hat{X}^\mu - v^\mu \hat{P}_\mu)} \,, \tag{A.1}$$

where $\mathcal{F}$ is the Fourier transform[10] of f over *whole* phase space (in other words, over position *and* momentum spaces)

$$\mathcal{F}(k, v) \;:=\; \frac{1}{(2\pi\hbar)^n} \int d^n x \, d^n p \; f(x, p) \, e^{-\frac{i}{\hbar} (k_\mu x^\mu - v^\mu p_\mu)} \,.$$

The function $f(x, p)$ is called the *Weyl symbol* of the operator $\hat{F}$, which need not be in symmetric-ordered form. A nice property of the Weyl map (A.1) is that it relates the complex conjugation $*$ of symbols to the Hermitian conjugation $\dagger$ of operators, $\mathcal{W} : f^*(x^\mu, p_\nu) \mapsto \hat{F}^\dagger$. Consequently, the image of a real function (a classical observable) is a Hermitian operator (a quantum observable). The inverse $\mathcal{W}^{-1} : \hat{F} \mapsto f(x^\mu, p_\nu)$ of the Weyl map is called the *Wigner map*.

The commutation relations between the position and momentum operators are $[\hat{X}^\mu, \hat{P}_\nu]_- = i\,\hbar\,\delta^\mu_\nu$, where $[\hat{A}, \hat{B}]_\pm := \hat{A}\hat{B} \pm \hat{B}\hat{A}$ denotes the (anti)commutator of the operators $\hat{A}$ and $\hat{B}$. The Baker-Campbell-Hausdorff formula implies that if the commutator $[\hat{A}, \hat{B}]_-$ itself commutes with both $\hat{A}$ and $\hat{B}$, then

$$e^{\hat{A}} \, e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}]_-} \,.$$

Moreover, for any operators $\hat{A}$ and $\hat{B}$ one can show that

$$e^{\hat{A}} \, e^{\hat{B}} \, e^{\pm\hat{A}} = e^{[\hat{A}, \ ]_\pm} \, e^{\hat{B}} \,,$$

where $[\hat{A}, \ ]_\pm$ denotes the (anti)adjoint action of $\hat{A}$. Two very useful equalities follow:

$$e^{\frac{i}{\hbar} (k_\mu \hat{X}^\mu - v^\mu \hat{P}_\mu)} \;=\; e^{\frac{i}{2\hbar} k_\mu \hat{X}^\mu} \, e^{-\frac{i}{\hbar} v^\mu \hat{P}_\mu} \, e^{\frac{i}{2\hbar} k_\mu \hat{X}^\mu} \tag{A.2}$$

$$=\; e^{\frac{i}{2\hbar} k_\mu [\hat{X}^\mu, \ ]_+} \, e^{-\frac{i}{\hbar} v^\mu \hat{P}_\mu} \tag{A.3}$$

Combining (A.1) with (A.3) implies that one way to explicitly perform the Weyl map is via some "anticommutator ordering" for half of the variables with respect to their conjugates.

The matrix elements in the position basis of the exponential operator in (A.1) are found to be equal to

$$\langle x \mid e^{\frac{i}{\hbar} (k_\mu \hat{X}^\mu - v^\mu \hat{P}_\mu)} \mid x' \rangle \;=\; e^{\frac{i}{2\hbar} k_\mu (x^\mu + x'^\mu)} \langle x \mid e^{-\frac{i}{\hbar} v^\mu \hat{P}_\mu} \mid x' \rangle$$

$$=\; \int \frac{d^n p}{(2\pi\hbar)^n} \, e^{\frac{i}{2\hbar} k_\mu (x^\mu + x'^\mu) + \frac{i}{\hbar} (x^\mu - x'^\mu - v^\mu) p_\mu} \tag{A.4}$$

---

[10]The Weyl map is well defined for a much larger class than square integrable functions, including for instance the polynomial functions (remark: their Fourier transform are distributions).

by making use of the identity (A.2) and by inserting the completeness relation $\int d^n p/(2\pi\hbar)^n \, |\, p \,\rangle\langle\, p\, | = \widehat{1}$.

The *integral kernel* of an operator $\hat{F}$ is the matrix element $\langle\, x \mid \hat{F} \mid x' \,\rangle$ appearing in the position representation of the state $\hat{F} \mid \psi \,\rangle$ as follows

$$\langle\, x \mid \hat{F} \mid \psi \,\rangle \;=\; \int d^n x' \; \psi(x') \, \langle\, x \mid \hat{F} \mid x' \,\rangle \,,$$

where the wave function in position space is $\psi(x') := \langle\, x' \mid \psi \,\rangle$ and the completeness relation $\int dx' \, |\, x' \,\rangle\langle\, x' \, | = \widehat{1}$ has been inserted. The definition (A.1) and the previous relation (A.4) enable to write the integral kernel of an operator in terms of its Weyl symbol,

$$\langle\, x \mid \hat{F} \mid x' \,\rangle \;=\; \int \frac{d^n p}{(2\pi\hbar)^n} \; f\left(\frac{x+x'}{2},\, p\right) \; e^{\frac{i}{\hbar}\,(x^\mu - x'^\mu)\,p_\mu} \,. \tag{A.5}$$

This provides an explicit form of the Wigner map

$$f(x^\mu, p_\nu) \;=\; \int d^n q \; \langle\, x - q/2 \mid \hat{F} \mid x + q/2 \,\rangle \; e^{\frac{i}{\hbar}\,q^\mu p_\mu} \,, \tag{A.6}$$

as follows from the expression (A.5). This shows that indeed the Weyl and Wigner maps are bijections between the vector spaces of classical and quantum observables. The Fourier transform

$$\breve{f}(x^\mu, v^\nu) \;:=\; \int \frac{d^n p}{(2\pi\hbar)^n} \; f(x^\mu, p_\nu) \; e^{\frac{i}{\hbar}\,v^\mu p_\mu} \,,$$

over momentum space of the Weyl symbol $f(x,p)$ is a function on the configuration space $T\mathbb{R}^n \cong \mathbb{R}^{2n}$. The equation (A.5) states that the Fourier transform over momentum space of the Weyl symbol is related to the integral kernel of its operator via

$$\langle\, x \mid \hat{F} \mid x' \,\rangle \;=\; \breve{f}\left(\frac{x+x'}{2},\, x^\mu - x'^\mu\right) \tag{A.7}$$

or, equivalently,

$$\breve{f}(x^\mu, v^\nu) \;=\; \langle\, x + v/2 \mid \hat{F} \mid x - v/2 \,\rangle \,. \tag{A.8}$$

By integrating over $x = x'$, the relation (A.5) also implies that the trace of an operator $\hat{F}$ is proportional to the integral over phase space of its Weyl symbol f,

$$\mathrm{Tr}[\hat{F}] \;=\; \frac{1}{(2\pi\hbar)^n} \int d^n x \, d^n p \; f(x,p) \,. \tag{A.9}$$

As a side remark, notice that the Fourier transform

$$\tilde{f}(k_\mu, p_\nu) \;:=\; \int d^n x \; f(x^\mu, p_\nu) \; e^{-\frac{i}{\hbar}\,k_\mu x^\mu} \,,$$

over position space of the Weyl symbol $f(x,p)$ is related to the matrix element in the momentum basis of the operator $\hat{F}$ via

$$\langle\, k \mid \hat{F} \mid k' \,\rangle \;=\; \tilde{f}\left(k^\mu - k'^\mu,\, \frac{k+k'}{2}\right) \tag{A.10}$$

in direct analogy with (A.7).

The *Moyal product* $\star$ is the pull-back of the composition product in the algebra of quantum observables with respect to the Weyl map $\mathcal{W}$, such that the latter becomes an isomorphism of associative algebras, namely

$$\mathcal{W}\big[f(x,p) \star g(x,p)\big] \, = \, \hat{F}\,\hat{G}\,. \tag{A.11}$$

The Wigner map (A.6) allows to check that the following explicit expression of the Moyal product satisfies the definition (A.11),

$$f(x,p) \star g(x,p) \, = \, f(x,p) \, \exp\left[\,\frac{i\,\hbar}{2}\,\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\,\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\,\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right]\,g(x,p)$$

$$= \, f(x,p)\,g(x,p) + \frac{i\,\hbar}{2}\,\{f(x,p)\,,\,g(x,p)\}_{\mathbf{P.B.}} + \mathcal{O}(\hbar^2) \tag{A.12}$$

where the arrows indicate on which factor the derivatives should act. The trace formula (A.9) for a product of operators leads to

$$\mathrm{Tr}[\,\hat{F}\hat{G}\,] \, = \, \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\,f(x,p) \star g(x,p)$$

$$= \, \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\,f(x,p)\,g(x,p) \tag{A.13}$$

because all terms in the Moyal product (A.12) beyond the pointwise product are divergences over phase space and any boundary term will always be assumed to be zero in the present notes.

The *Wigner function* $\rho(x,p)$ is the Weyl symbol of the density operator $\hat{\rho}$ under the Wigner map (A.6). Let $|\,\psi\,\rangle$ be an (unnormalised) quantum state. The corresponding pure state density operator is equal to $\hat{\rho} := |\,\psi\,\rangle\langle\,\psi\,|$. Then the Fourier transform over momentum space of the pure state Wigner function $\rho(x,p)$ can be written in terms of the wave function $\psi(x)$ as follows,

$$\breve{\rho}(x,q) \, = \, \psi(x+q/2)\,\psi^*(x-q/2)\,, \tag{A.14}$$

due to (A.8). The mean value of an observable $\hat{F}$ over the state $|\,\psi\,\rangle$ is proportional to the integral over phase space of the product between the Wigner function ρ and the Weyl symbol f,

$$\langle\,F\,\rangle_\psi \, = \, \frac{\langle\,\psi\,|\,\hat{F}\,|\,\psi\,\rangle}{\langle\,\psi\,|\,\psi\,\rangle} \, = \, \frac{\mathrm{Tr}[\,\hat{\rho}\,\hat{F}\,]}{\mathrm{Tr}\,[\,\hat{\rho}\,]} \, = \, \frac{\int dx\,dp\,\,\rho(x,p)\,f(x,p)}{\int dx\,dp\,\,\rho(x,p)}\,, \tag{A.15}$$

which explains why the Wigner function is sometimes called the Wigner "quasi-probability distribution." It should be emphasised that the Wigner function is real but may take negative values, thereby exhibiting quantum behaviour.

Let $\hat{H}$ be a Hamiltonian operator of Weyl symbol $h(x,p)$. In the Heisenberg picture, the time evolution of quantum observables (which do not depend explicitly on time) is governed by the differential equation

$$\frac{d\hat{F}}{dt} \, = \, \frac{1}{i\,\hbar}\,[\hat{F},\hat{H}]_- \quad \Longleftrightarrow \quad \frac{df}{dt} \, = \, \frac{1}{i\,\hbar}\,[\,f^\star_,h\,]_- \tag{A.16}$$

where $[\ \overset{\star}{,}\ ]_-$ denotes the *Moyal commutator* defined by

$$
\begin{aligned}
[\, f(x,p) \overset{\star}{,} g(x,p) \,]_- &:= f(x,p) \star g(x,p) \, - \, g(x,p) \star f(x,p) \\
&= 2i \, f(x,p) \, \sin\left[\frac{\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu} \frac{\overrightarrow{\partial}}{\partial x^\mu} \right) \right] g(x,p) \\
&= i\hbar \, \{ f(x,p), g(x,p) \}_{\textbf{P.B.}} + \mathcal{O}(\hbar^2),
\end{aligned} \tag{A.17}
$$

as can be seen from (A.12). essentially to the Poisson bracket. The *Moyal bracket* is the renormalisation of the Moyal commutator given by

$$
\frac{1}{i\hbar} [\ \overset{\star}{,}\ ]_- = \{\ ,\ \}_{\textbf{P.B.}} + \mathcal{O}(\hbar).
$$

The Moyal bracket is a deformation of the Poisson bracket, and one can see that the equation (A.16) in terms of the Weyl symbol is a perturbation of the Hamiltonian flow. If either $f(x,p)$ or $g(x,p)$ is a polynomial of degree two, then their Moyal bracket reduces to their Poisson bracket. So when the Hamiltonian is quadratic (free) the quantum evolution of a Weyl symbol is identical to its classical evolution.

B Elastic scattering

The three *Mandelstam variables* s, t and u of any elastic scattering of four particles (see e.g. the textbook [31]) with the same mass m are related by $s + t + u = 4 m^2$. In $n = 4$ dimensions, there are indeed only two independent Lorentz invariants which can be constructed from the four 4-momenta.

Let the Mandelstam variables of the scattering $\phi(k_1) \phi(k_2) \to \phi(\ell_1) \phi(\ell_2)$ be

$$
s = -(k_1 + k_2)^2, \qquad t = -(k_1 - \ell_1)^2, \qquad u = -(k_1 - \ell_2)^2. \tag{B.1}
$$

In the center-of-mass system, the four-momenta take the form

$$
k_1^\mu = \left(\frac{\sqrt{s}}{2}, \vec{k} \right), \qquad k_2^\mu = \left(\frac{\sqrt{s}}{2}, -\vec{k} \right), \qquad \ell_1^\mu = \left(\frac{\sqrt{s}}{2}, \vec{\ell} \right), \qquad \ell_2^\mu = \left(\frac{\sqrt{s}}{2}, -\vec{\ell} \right),
$$

hence the variable $s \geqslant (2m)^2$ is the *squared center of mass energy*, the variable $t = -(\vec{k} - \vec{\ell})^2$ is the *squared momentum transfer* and $u = -(\vec{k} + \vec{\ell})^2$ has no obvious physical interpretation. The (center-of-mass) *scattering angle* θ is defined as the angle between $\vec{k}$ and $\vec{\ell}$. The products of momenta are related by

$$
k_1 \cdot k_2 = \ell_1 \cdot \ell_2 = m^2 - \frac{s}{2}, \qquad\qquad k_1 \cdot \ell_1 = k_2 \cdot \ell_2 = \frac{t}{2} - m^2,
$$

$$
k_1 \cdot \ell_2 = k_2 \cdot \ell_1 = \frac{u}{2} - m^2.
$$

The two relevant variables of the problem considered in the paper are

$$
s + u = -(k_1 + \ell_1)^2 = -(k_2 + \ell_2)^2, \qquad s - u = -(k_1 + \ell_1) \cdot (k_2 + \ell_2).
$$

Both variables can be expressed in terms of the squared center-of-mass energy s and scattering angle θ as

$$s \pm u = \left[1 \mp \cos^2\left(\frac{\theta}{2}\right) \right] s \pm 4m^2 \,.$$

Hence, for large $s \gg m^2$, they behave in the Regge limit[11] as

$$s + u \sim -t \quad \text{is fixed and} \quad \frac{s-u}{s+u} \sim -\frac{2}{t}\, s \quad \text{is large}\,,$$

and in the fixed scattering angle limit as

$$\frac{s-u}{s+u} \sim \frac{1 + \cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \quad \text{is fixed and} \quad s + u \sim \sin^2\left(\frac{\theta}{2}\right) s \quad \text{is large}\,.$$

In the scattering theory of quantum mechanics, the differential cross section between two boson is given by

$$\frac{d\sigma}{d\Omega} = \left| f(\vec{k}, \vec{\ell}\,) + f(\vec{k}, -\vec{\ell}\,) \right|^2 \,. \tag{B.2}$$

and in the *Born approximation* the scattered waves f are proportional to the Fourier transform of the potential $V(\vec{x}\,)$:

$$f(\vec{k}, \vec{\ell}\,) = -\frac{m}{4\pi} \int d^3x\, V(\vec{x}\,)\, e^{i\,(\vec{k}-\vec{\ell}\,)\cdot\vec{x}} \,. \tag{B.3}$$

Comparing eq. (B.2) with the differential cross section calculated from the scattering amplitude $A(s,t,u)$:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\,\pi^2\,s} \left| A(s,t,u) + A(s,u,t) \right|^2 \,, \tag{B.4}$$

we get the non-relativistic interaction potential as

$$V(\vec{x}\,) = -\frac{1}{4\,m^2} \int \frac{d^3p}{(2\pi)^3}\, A\left(4m^2, -\vec{p}^2, 0\right)\, e^{-i\,\vec{p}\cdot\vec{x}} \,. \tag{B.5}$$

Remark that another way[12] of deriving the low-energy interaction potentials is by considering a distribution

$$\overset{(r)}{J}{}^{\mu_1 \cdots \mu_r}(x) \;=\; \sigma(x)\, w^{\mu_1} \ldots w^{\mu_r} \tag{B.6}$$

of particles at rest of density and velocity respectively given by the scalar $\sigma(\vec{x})$ and the fixed vector w^μ. Plugging (B.6) inside the integrals (4.4) leads to a current-current interaction in $n = 4$ given by

$$\begin{aligned}
\overset{(r)}{S}{}_{\text{curr}}\,[\,\sigma\,] &= \int dx^0 \int d^3x\, \sigma(\vec{x})\, \frac{1}{\Delta}\, \sigma(\vec{x})\,, \\
&= \int dx^0 \int d^3x\, d^3y\, \sigma(\vec{x})\, \sigma(\vec{y})\, \overset{(r)}{V}(\rho)\,,
\end{aligned} \tag{B.7}$$

where $\overset{(r)}{V}(\rho)$ is the interaction potential computed in (6.2) where $m\lambda = 1$.

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[11]Traditionnally, one distinguishes two high-energy (i.e. $s/m^2 \to \infty$) limits: the *Regge* (or *fixed momentum transfer*) limit which corresponds to $s/m^2 \to \infty$ with t fixed (thus $\theta \to 0$ and $u/m^2 \to -\infty$) and the *fixed scattering angle* limit which corresponds to $s/m^2 \to \infty$ with s/t and u/s fixed (thus $t/m^2 \to -\infty$ and $u/m^2 \to -\infty$).

[12]See e.g. [32] for the case of massive mediating fields.

C Chebyshev and Gegenbauer polynomials

Several useful definitions and formulas taken from [33] are collected here in order to be self-contained.

The *Chebyshev polynomial of first kind* $T_r(z)$ is uniquely defined by the relation $T_r(\cos \beta) = \cos(r\beta)$ for any angle β, which implies

$$T_r(z) = \frac{1}{2} \left[\left(z + \sqrt{z^2 - 1} \right)^r + \left(z - \sqrt{z^2 - 1} \right)^r \right] \tag{C.1}$$

but it is also given by the sum

$$T_r(z) = \frac{r}{2} \sum_{m=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^m (r - m - 1)!}{m!\,(r - 2m)!} \, (2z)^{r-2m} \,, \tag{C.2}$$

when $r \geqslant 1$. Observe that $T_0(z) = 1 = T_r(1)$ and $T_r(-z) = (-1)^r T_r(z)$. When $|z| \gg 1$, the Chebyshev polynomial of first kind with index $r \geqslant 1$ behaves as $T_r(z) \sim 2^{r-1}\, r\, z^r$.

The *Gegenbauer (or ultraspherical) polynomial* $C_r^\alpha(z)$ with $\alpha > -\frac{1}{2}$ et $\alpha \neq 0$ is a polynomial of degree $r \in \mathbb{N}$ in the variable z defined as

$$C_r^\alpha(z) := \sum_{m=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^m (\alpha)_{r-m}}{m!\,(r - 2m)!} \, (2z)^{r-2m} \,. \tag{C.3}$$

They generalize the Legendre polynomials $P_r(z)$ to which $C_r^{\frac{1}{2}}(z)$ is proportional. Moreover, the Chebyshev polynomial of first kind $T_r(z)$ may somehow be thought as a regularised limit of Gegenbauer polynomials $C_r^\alpha(z)$ for $\alpha \to 0$. Notice that $C_0^\alpha(z) = 1$ and $C_r^\alpha(-z) = (-1)^r C_r^\alpha(z)$. When $|z| \gg 1$, the Gegenbauer polynomial behaves as $C_r^\alpha(z) \sim (\alpha)_r (2z)^r / r!$.

References

[1] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in* $(3 + 1)$-*dimensions*, *Phys. Lett.* **B 243** (1990) 378 [SPIRES]; *Properties of equations of motion of interacting gauge fields of all spins in* $(3 + 1)$-*dimensions*, *Class. Quant. Grav.* **8** (1991) 1387 [SPIRES]; *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B 285** (1992) 225 [SPIRES]; *Nonlinear equations for symmetric massless higher spin fields in* $(A)dS_d$, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049] [SPIRES].

[2] M.A. Vasiliev, *Higher spin gauge theories in various dimensions*, *Fortsch. Phys.* **52** (2004) 702 [hep-th/0401177] [SPIRES]; *Higher spin gauge theories in any dimension*, *Comptes Rendus Physique* **5** (2004) 1101 [hep-th/0409260] [SPIRES]; X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, in the Proceedings of the *First Solvay workshop on higher-spin gauge theories*, Brussels Belgium (2004), hep-th/0503128 [SPIRES].

[3] S. Weinberg, *Photons and gravitons in S matrix theory: derivation of charge conservation and equality of gravitational and inertial mass*, *Phys. Rev.* **135** (1964) B1049 [SPIRES]; M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, *Supergravity and the S matrix*, *Phys. Rev.* **D 15** (1977) 996 [SPIRES];

Pedagogical review in S. Weinberg, *The quantum theory of fields. Volume I: Foundations*, Cambridge University Press, Cambridge U.K. (1995), section 13.1.

[4] S.R. Coleman and J. Mandula, *All possible symmetries of the S matrix*, *Phys. Rev.* **159** (1967) 1251 [SPIRES];
R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, *Nucl. Phys.* **B 88** (1975) 257 [SPIRES];
Pedagogical review in S. Weinberg, *The quantum theory of fields. Volume III: Supersymmetry*, Cambridge University Press, Cambridge U.K. (2000), chapter 24.

[5] C. Aragone and S. Deser, *Consistency problems of hypergravity*, *Phys. Lett.* **B 86** (1979) 161 [SPIRES];
F.A. Berends, J.W. van Holten, B. de Wit and P. van Nieuwenhuizen, *On spin 5/2 gauge fields*, *J. Phys.* **A 13** (1980) 1643 [SPIRES];
C. Aragone and H. La Roche, *Massless second order tetradic spin 3 fields and higher helicity bosons*, *Nuovo Cim.* **A 72** (1982) 149 [SPIRES];
S. Deser and Z. Yang, *Inconsistency of spin 4 – spin 2 gauge field couplings*, *Class. Quant. Grav.* **7** (1990) 1491 [SPIRES];
M. Porrati, *Universal limits on massless high-spin particles*, *Phys. Rev.* **D 78** (2008) 065016 [arXiv:0804.4672] [SPIRES].

[6] N. Boulanger and S. Leclercq, *Consistent couplings between spin-2 and spin-3 massless fields*, *JHEP* **11** (2006) 034 [hep-th/0609221] [SPIRES];
N. Boulanger, S. Leclercq and P. Sundell, *On the uniqueness of minimal coupling in higher-spin gauge theory*, *JHEP* **08** (2008) 056 [arXiv:0805.2764] [SPIRES].

[7] A.Y. Segal, *Conformal higher spin theory*, *Nucl. Phys.* **B 664** (2003) 59 [hep-th/0207212] [SPIRES].

[8] M.A. Vasiliev, *Extended higher spin superalgebras and their realizations in terms of quantum operators*, *Fortsch. Phys.* **36** (1988) 33 [SPIRES].

[9] D. Francia, J. Mourad and A. Sagnotti, *Current exchanges and unconstrained higher spins*, *Nucl. Phys.* **B 773** (2007) 203 [hep-th/0701163] [SPIRES].

[10] H. Weyl, *Quantum mechanics and group theory*, *Z. Phys.* **46** (1927) 1 [SPIRES];
E.P. Wigner, *On the quantum correction for thermodynamic equilibrium*, *Phys. Rev.* **40** (1932) 749 [SPIRES];
J.E. Moyal, *Quantum mechanics as a statistical theory*, *Proc. Cambridge Phil. Soc.* **45** (1949) 99 [SPIRES];
M. Hillery, R.F. O'Connell, M.O. Scully and E.P. Wigner, *Distribution functions in physics: fundamentals*, *Phys. Rept.* **106** (1984) 121 [SPIRES].

[11] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev.* **D 18** (1978) 3624 [SPIRES].

[12] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, *Nucl. Phys.* **B 633** (2002) 3 [hep-th/0111246] [SPIRES].

[13] F.A. Berends, G.J.H. Burgers and H. van Dam, *Explicit construction of conserved currents for massless fields of arbitrary spin*, *Nucl. Phys.* **B 271** (1986) 429 [SPIRES].

[14] D. Anselmi, *Higher-spin current multiplets in operator-product expansions*, *Class. Quant. Grav.* **17** (2000) 1383 [hep-th/9906167] [SPIRES];
M.A. Vasiliev, *Higher spin gauge theories: star-product and AdS space*, hep-th/9910096 [SPIRES];

S.E. Konstein, M.A. Vasiliev and V.N. Zaikin, *Conformal higher spin currents in any dimension and AdS/CFT correspondence*, *JHEP* **12** (2000) 018 [`hep-th/0010239`] [SPIRES];
O.A. Gelfond, E.D. Skvortsov and M.A. Vasiliev, *Higher spin conformal currents in Minkowski space*, *Theor. Math. Phys.* **154** (2008) 294 [`hep-th/0601106`] [SPIRES].

[15] R. Manvelyan and W. Rühl, *Conformal coupling of higher spin gauge fields to a scalar field in AdS_4 and generalized Weyl invariance*, *Phys. Lett.* **B 593** (2004) 253 [`hep-th/0403241`] [SPIRES];
A. Fotopoulos, N. Irges, A.C. Petkou and M. Tsulaia, *Higher-spin gauge fields interacting with scalars: the Lagrangian cubic vertex*, *JHEP* **10** (2007) 021 [`arXiv:0708.1399`] [SPIRES];
R. Manvelyan and K. Mkrtchyan, *Conformal invariant interaction of a scalar field with the higher spin field in AdS_D*, `arXiv:0903.0058` [SPIRES].

[16] X. Bekaert, *Higher spin algebras as higher symmetries*, in the Proceedings of the *Fifth international spring school and workshop on quantum field theory and Hamiltonian systems*, Calimanesti & Caciulata Romania (2006), `arXiv:0704.0898` [SPIRES].

[17] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory, Vol. 1: Introduction*, Cambridge University Press, Cambridge U.K. (1987), chapter 1.

[18] M. Grigoriev, *Off-shell gauge fields from BRST quantization*, `hep-th/0605089` [SPIRES].

[19] X. Bekaert, *Comments on higher-spin symmetries*, *Int. J. Geom. Meth. Mod. Phys.* **6** (2009) 285 [`arXiv:0807.4223`] [SPIRES].

[20] M.A. Vasiliev, *Actions, charges and off-shell fields in the unfolded dynamics approach*, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 37 [`hep-th/0504090`] [SPIRES].

[21] E. Witten, *Baryons and branes in anti de Sitter space*, talk at *Strings '98*, Santa Barbara CA U.S.A. (June 1998), http://online.itp.ucsb.edu/online/strings98/witten.

[22] B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 113 [`hep-th/0103247`] [SPIRES];
E. Witten, *Spacetime reconstruction*, talk at *J.H. Schwarz* 60-*th Birthday Conference*, CalTech CA U.S.A. (November 2001), http://theory.caltech.edu/jhs60/witten/1.html;
E. Sezgin and P. Sundell, *Doubletons and 5D higher spin gauge theory*, *JHEP* **09** (2001) 036 [`hep-th/0105001`] [SPIRES];
I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett.* **B 550** (2002) 213 [`hep-th/0210114`] [SPIRES].

[23] D. Francia and A. Sagnotti, *On the geometry of higher-spin gauge fields*, *Class. Quant. Grav.* **20** (2003) S473 [`hep-th/0212185`] [SPIRES]; *Higher-spin geometry and string theory*, *J. Phys. Conf. Ser.* **33** (2006) 57 [`hep-th/0601199`] [SPIRES].

[24] I.L. Buchbinder, A.V. Galajinsky and V.A. Krykhtin, *Quartet unconstrained formulation for massless higher spin fields*, *Nucl. Phys.* **B 779** (2007) 155 [`hep-th/0702161`] [SPIRES].

[25] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, *Parent field theory and unfolding in BRST first-quantized terms*, *Commun. Math. Phys.* **260** (2005) 147 [`hep-th/0406192`] [SPIRES].

[26] B.V. Fedosov, *Quantization and the index Soviet Phys. Dokl.* **31** (1986) 877 [*Dokl. Akad. Nauk SSSR* **291** (1986) 82]; *A simple geometrical construction of deformation quantization*, *J. Diff. Geom.* **40** (1994) 213 [SPIRES].

[27] M. Bordemann, N. Neumaier and S. Waldmann, *Homogeneous Fedosov star products on cotangent bundles I: Weyl and standard ordering with differential operator representation*, q-alg/9707030 [SPIRES]; *Homogeneous Fedosov star products on cotangent bundles II: gNS representations, the WKB expansion and applications*, q-alg/9711016 [SPIRES].

[28] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong* $\mathrm{Sp}(2, R)$ *condition*, in the Proceedings of the *First Solvay workshop on higher-spin gauge theories*, Brussels Belgium (May 2004), hep-th/0501156 [SPIRES].

[29] D.P. Sorokin and M.A. Vasiliev, *Reducible higher-spin multiplets in flat and AdS spaces and their geometric frame-like formulation*, *Nucl. Phys.* **B 809** (2009) 110 [arXiv:0807.0206] [SPIRES].

[30] X. Bekaert, E. Joung, J. Mourad, *Weyl calculus and Noether currents: an application to cubic interactions*, *Physics Ann. Univ. Craiova* **18** (2008) 26.

[31] R. Hagedorn, *Relativistic kinematics*, W.A. Benjamin, New York U.S.A. (1963) chapter 6.

[32] K. Jagannathan and L.P.S. Singh, *Attraction/repulsion between like charges and the spin of the classical mediating field*, *Phys. Rev.* **D 33** (1986) 2475 [SPIRES].

[33] M. Abramowitz and I.A. Stegun eds., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover, New York U.S.A. (1972), chapter 22;
S. Iyanaga and Y. Kawada eds., *Encyclopedic dictionary of mathematics*, MIT Press (1980), appendix A, table 20.I.